30/04/2018

1.) $f:[\alpha,\beta] \to \mathbb{R}$

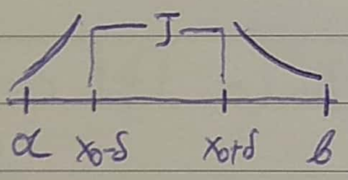
συνχ. στο $x_0 \in (\alpha,\beta)$ $\quad$ f: $\mathbb{R}$-ολοκλ.

$$\left.\begin{array}{l} f(x) \geq 0, \ \forall x \\ f(x_0) > 0 \end{array}\right\} \Rightarrow \int_\alpha^\beta f > 0$$

$$\left.\begin{array}{l} f(x_0) > 0 \\ f \text{ συνχ. στο } x_0 \end{array}\right\} \Rightarrow \begin{array}{l} \exists\, \delta > 0 \ \text{ ώστε } \ \underbrace{[x_0-\delta, x_0+\delta]}_{= J} \subseteq [\alpha,\beta] \\ \exists\, \underbrace{c_{x_0,\delta}}_{= c} > 0 : f(x) \geq c, \ \forall x \in J \end{array}$$

Μπορείς να διαλέξεις $\quad c = \dfrac{f(x_0)}{2}$

f ολοκλ. στο $[\alpha,\beta]$

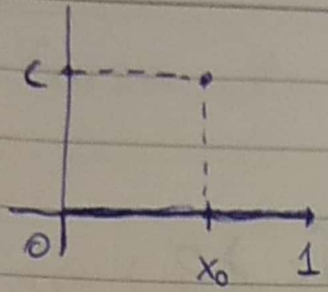


$$\int_\alpha^\beta f = \int_\alpha^{x_0-\delta} f + \int_J f + \int_{x_0+\delta}^\beta f = I_1 + I_2 + I_3 \geq 2\delta c > 0$$

$I_1 \geq 0(x_0-\delta-\alpha) = 0$

$I_3 \geq 0(\beta-(x_0+\delta)) = 0$

$I_2 \geq c(x_0+\delta-(x_0-\delta)) = 2\delta \cdot c$



$f(x)=0, \quad \forall x \neq x_0$

$f(x_0)=c>0$

1) (;) f: R-ολοκλ.

2) Αν ναι $\int_0^1 f =$;

$\forall \varepsilon>0$, $f/[0, x_0-\varepsilon]$ είναι ολοκλ. σαν συνεχ. $\Rightarrow$ $f|[0, x_0-\varepsilon]$ R-ολοκλ.

Παρόμοια $f/[x_0, 1]$ R-ολοκλ.

$\Rightarrow$ f R-ολοκλ. στο $[0,1]$

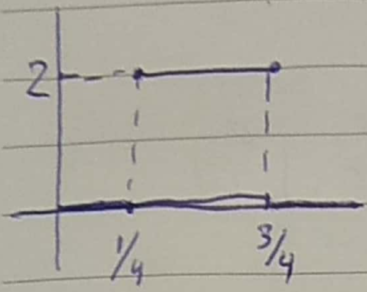
$\Rightarrow \int_0^1 f = \int_0^{x_0} f + \int_{x_0}^1 f$

Αρκεί ν.δ.ο. $0 \leq \int_0^1 f \leq [c \cdot \varepsilon] \quad \forall \varepsilon>0, \forall c \in \mathbb{R}>0$

$$\int_0^1 f = \cancel{\int_0^{x_0-\frac{\varepsilon}{2}} f}^{\,0} + \int_{x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} f + \cancel{\int_{x_0+\frac{\varepsilon}{2}}^{1} f}^{\,0}$$

$$\int_0^1 f = \int_{x_0-\frac{\varepsilon}{2}}^{x_0+\varepsilon/2} f \leq \underset{\downarrow\, \varepsilon \to 0^+ \atop 0}{c \cdot \varepsilon}, \quad \forall \varepsilon>0$$

$$\Rightarrow \int_0^1 f = 0$$



$$\int_0^1 f = \cancel{\int_0^{1/4} f}^{\,0} + \int_{1/4}^{3/4} f + \cancel{\int_{3/4}^{1} f}^{\,0}$$

$$= I_1 + \frac{1}{2}\cdot 2 + I_3 = 0 + 1 + 0 = 1$$

$f: R$-ολοκλ. $\int_0^1 f = ;$

$$0 \leq I_1 = \int_0^{1/4} f = \int_0^{1/4-\varepsilon} f + \int_{1/4-\varepsilon}^{1/4} f = 0 + \int_{1/4-\varepsilon}^{1/4} f \leq 2\left(\frac{1}{4} - \left(\frac{1}{4} - \varepsilon\right)\right) = 2\varepsilon \to 0 \quad \forall \varepsilon > 0$$

Πρόταση: Δίνεται $f : [\alpha, \beta] \xrightarrow{\text{ολ}} [m, M]$ ολοκληρ.

$$\begin{array}{ccc} [\alpha,\beta] & \xrightarrow{f} & [m,M] \\ & \searrow^{\varphi\circ f} & \downarrow \varphi \ (\text{συνεχ.}) \\ & & \mathbb{R} \end{array}$$

Τότε $\varphi \circ f$ R-ολοκλ.

Εφαρμογές

$[\alpha, \beta] \xrightarrow{f} \mathbb{R}$ - ολοκλ.

$\Rightarrow |f|$ ολοκλ. στο $[\alpha, \beta]$

$f, g : [\alpha, \beta] \to \mathbb{R}$ ολοκλ.

$\Rightarrow f \cdot g$ ολοκλ.;

$$f \cdot g = \frac{(f+g)^2 - (f-g)^2}{4} \Rightarrow f \cdot g \text{ ολοκλ.}$$

Πρoτ.
$f:[\alpha,\beta]\to\mathbb{R}$ ολοκλ.
$\Rightarrow |f|$ ολοκλ. $\left|\int_\alpha^\beta f\right| \le \int_\alpha^\beta |f| \iff -\int_\alpha^\beta |f| \le \int_\alpha^\beta f \le \int_\alpha^\beta |f|$

αν $f(x)\le g(x)$, $\forall x\in[\alpha,\beta] \Rightarrow \int_\alpha^\beta f \le \int_\alpha^\beta g$

παρατ. ότι $-|f(x)|\le f(x)\le |f(x)|$, $\forall x\in[\alpha,\beta]$

$$\Rightarrow \int_\alpha^\beta -|f(x)|\,dx \le \int_\alpha^\beta f(x)\,dx \le \int_\alpha^\beta |f(x)|\,dx = \int_\alpha^\beta |f|$$

Πρoτ. $f:[\alpha,\beta]\to\mathbb{R}$ συνx (άρα ολοκλ)
Τότε $\exists\,\xi\in[\alpha,\beta]$ ώστε

$$f(\xi)=\frac{1}{\beta-\alpha}\int_\alpha^\beta f \quad ① \iff (\beta-\alpha)\,f(\xi)$$

Αποδ. f συνx στο $[\alpha,\beta]$ $m=\inf\{f(x):x\in[\alpha,\beta]\}=\min\{f(x):x\in[\alpha,\beta]\}$
$M=\sup f=\max f$

$\Rightarrow m\le f(x)\le M=\max f$

$$\Rightarrow m\int_\alpha^\beta dx \le \int_\alpha^\beta f(x)\,dx \le M\int_\alpha^\beta dx$$

$$\|$$

$$m(\beta-\alpha)\le \int_\alpha^\beta f \le M(\beta-\alpha)$$

$$f(x_0)=m\le \frac{\int_\alpha^\beta f}{\beta-\alpha}\le M \Rightarrow m\le H\le M$$

Θεωρούμε

$f:[\alpha,b]\to\mathbb{R}$ R-ολοκλ. κατασκευάζουμε την $F:[\alpha,b]\to\mathbb{R}$

με τύπο $F(x)=\int_\alpha^x f(t)dt$, $x\in[\alpha,b]$

$x=\alpha$ $\int_\alpha^\alpha F(t)dt=0$

$x=b$, $F(b)=\int_\alpha^b f$

Πρόταση

f: R-ολοκλ. $\Rightarrow F$ συνχ.

Αποδ.

f φραγμ. $\Rightarrow \exists M>0: |f(x)|\leq M$, $\forall x\in[\alpha,b]$

Έστω $\alpha<x<y<b$

$|F(y)-F(x)|=\left|\int_\alpha^y f-\int_\alpha^x f\right|=\left|\int_x^y f\right|\leq\left|\int_x^y |f|(t)dt\right|=$

$=\int_x^y |f(t)|\,dt\leq\int_x^y M\,dt=M(y-x)=M|y-x|$.

$\Rightarrow F\ Lip(M)\Rightarrow F$ συνχ

Πρόταση

$f:[\alpha,b]\to\mathbb{R}$ R-ολοκλ. στο $[\alpha,b]$ συνχ στο $[\alpha,b]$

κ $F(x)=\int_\alpha^x f(t)\,dt$, Τότε $\exists F'(x_0)=f(x_0)$

Ορ. $G[\alpha,b]\to\mathbb{R}$ θα λέγεται παραγωγ. στο $[\alpha,b]$

αν 1) $\forall x\in(\alpha,b)\ \exists\ G'(x)=\lim_{h\to 0}\frac{G(x+h)-G(x)}{h}$

2) $\exists\, G_+'(\alpha) = \lim\limits_{x\to 0^+} \dfrac{G(\alpha+h) - G(\alpha)}{h}$

Αποδ. Θεωρ.

Χ.β.τ.γ. υποθ. ότι $x_0 \in (\alpha, \beta)$

Διαλέγουμε $\delta_1 > 0 : (x_0-\delta_1, x_0+\delta_1) \subseteq (\alpha, \beta)$

$\delta_1 = \min\{x_0-\alpha, \beta - x_0\}$

Θ.δ.ο. $F'(x_0) = f(x_0)$

$$\Lambda = \frac{F(x_0+h) - F(x_0)}{h} - f(x_0) = \frac{\int_\alpha^{x_0+h} f - \int_\alpha^{x_0} f}{h} - f(x_0) =$$

$\shortparallel$

$\Lambda(h)$

Θ.δ.ο. $\lim\limits_{h\to 0} \Lambda(h) = 0$

$h>0$

$$\frac{\int_{x_0}^{x_0+h} f}{h} - f(x_0)$$

$h<0$

$$\frac{-\int_{x_0+h}^{x_0} f}{h} - f(x_0)$$

$$= \frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - f(x_0)$$

$$\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\,dt = \frac{1}{h}\int_{x_0}^{x_0+h} f(t) - f(x_0)\,dt.$$

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$$|\Lambda(h)| = \left|\frac{1}{h}\int_{x_0}^{x_0+h}[f(t)-f(x_0)]\,dt\right| = \frac{1}{|h|}\left|\int_{x_0}^{x_0+\delta}[f(t)-F(x_0)]dt\right| =$$

Αν $\varepsilon>0$, $\exists\, \underset{0}{\overset{\vee}{\delta}} < \delta_1 : |f(t)-f(x_0)|<\varepsilon \quad \forall t \in (x_0-\delta, x_0+\delta)$

$h>0$

$$\leq \frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)\,dt \leq \frac{1}{h}\int_{x_0}^{x_0+h}\varepsilon\,dt = \varepsilon,\ \forall h\in(0,\delta)$$

$$h<0 \quad \frac{1}{-h}\left|\int_{x_0}^{x_0+h}(f(t)-f(x_0))\,dt\right| = -\frac{1}{h}\left|\int_{x_0+h}^{x_0}\Big[f(t)-f(x_0)\Big]dt\right|$$

$$\leq \frac{1}{-h}\int_{x_0+h}^{x_0}|f(t)-f(x_0)|\,dt \leq \frac{1}{-h}\varepsilon(-h) = \varepsilon.$$

Πόρισμα

$f:[\alpha,b]\to\mathbb{R}$ συνεχ. στο $[\alpha,b]$

και $F(x)=\int_{\alpha}^{x} f,\ x\in[\alpha,b]$

$\Rightarrow F$ παραγωγ. στο $[\alpha,b]$ και $F'=f$

$$H = \frac{1}{b-\alpha}\int_{\alpha}^{b} f = \frac{1}{b-\alpha}(F(b)-F(\alpha)) = \frac{F(b)-F(\alpha)}{b-\alpha} = \underset{f(3)}{F'(3)}$$

Αν f συνx. στο $[\alpha, \beta] \Rightarrow F' = f$ όπου $F(x) = \int_\alpha^x f$

Εστω G παράγουσα της $f \Rightarrow G' = f \Rightarrow (G-F)' = f - f = 0$

$\Downarrow$

$G - F =$ σταθ.

$G(x) - F(x) = c$

$\forall x \in [\alpha, \beta]$

$G(x) = \int_\alpha^x f(t)\,dt + c \xrightarrow{x=\alpha}$

$G(x) - G(a) = \int_\alpha^x f = \int_\alpha^x G'(t)\,dt$

Δ.ο. αν $G : [\alpha, \beta] \to \mathbb{R}$

$\exists G'$: και είναι συνx.

$\int_\alpha^x G'(t)\,dt = G(x) - G(a)$

Ασκ. $\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) - \int_1^n \frac{1}{x}\,dx =$

$= \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\,dx \to \gamma$

$f(x) = \frac{1}{x}$, $x \geq 1$ Δ.ο. $\frac{1}{k+1} \leq \int_k^{k+1} \frac{dx}{x} \leq \frac{1}{k}$

διότι $\forall x \in [k, k+1] \cdot \frac{1}{k+1} \leq f(x) \leq \frac{1}{k}$

$$\Rightarrow \frac{1}{k+1} \leq \int_k^{k+1} f \leq \frac{1}{k}(k+1-k) = \frac{1}{k}$$

$$\gamma_{n+1} - \gamma_n = \frac{1}{n-1} - \int_n^{n+1} \frac{dx}{x} \leq 0 \Rightarrow (\gamma_n)_n \downarrow$$

$$\gamma_n = \frac{1}{n} + \left(1 - \int_1^2 \frac{dx}{x}\right) + \left(\frac{1}{2} - \int_2^3 \frac{dx}{x}\right) + \dots + \left(\frac{1}{(n-1)} - \int_{n-1}^{n} \frac{dx}{x}\right) >$$

$$> \frac{1}{n} + 0 = \frac{1}{n}$$

$$= \frac{1}{n} + \left[\sum_{k=1}^{n-1} \left(\frac{1}{k} - \int_k^{k+1} \frac{dx}{x}\right)\right] \Rightarrow \sum_{k=1}^{\infty} \left(\frac{1}{k} - \int_k^{k+1} \frac{dx}{x}\right) < +\infty$$